Approximability, compactness and random dense sequences

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Notation

X separable metric space
$\mathcal{R}_X := \{\rho \,\big|\, \rho :\subseteq \mathbb{N}^{\mathbb{N}} \to X \text{ onto}\}$
$\mathcal{I}_{\mathsf{d}} := \{\nu \,\big|\, \nu :\subseteq \mathbb{N} \to X \text{ dense}\}$

Fix open cover $(U_i)_1^r$, $[r] := \{1, \ldots, r\}$
$R : X \rightrightarrows [r], x \mapsto \{i \,\big|\, U_i \ni x\}$
$S : [r] \rightrightarrows X, i \mapsto U_i$
$T_\epsilon : X \rightrightarrows X, x \mapsto \{y \,\big|\, d(x,y) < \epsilon\}$ $(\epsilon > 0)$

$\delta \leq_{\mathsf{a}} \rho :\iff (\forall \epsilon > 0)\,(T_\epsilon$ is (δ, ρ)-computable $)$
$\delta \leq \rho \iff (\mathrm{id}_X : X \to X$ is (δ, ρ)-computable $)$

$\delta \in \mathcal{R}_X$ **compact** if for each finite open cover $(U_i)_1^r$, $R : X \rightrightarrows [r]$ is $(\delta, \delta_{\mathbb{N}}|^{[r]})$-computable

Example: If X compact and $\nu \in \mathcal{I}_{\mathrm{d}}$, the *standard representation* (equivalent to Cauchy representation in case of a computable metric space) defined by

$$p \in \delta^{-1}\{x\} :\iff \{p_i - 1 \,\big|\, i \in \mathbb{N} \wedge p_i \geq 1\} = \{k \,\big|\, \alpha(k) \ni x\},$$

$\alpha : \mathbb{N} \to \mathcal{T}_X, \langle i, j\rangle \mapsto B_d(\nu(i); \nu_{\mathbb{Q}^+}(j))$.

Lemma 1. *Suppose X compact, $\delta, \rho \in \mathcal{R}_X$, δ compact. If ρ-computable points are dense in X then $\delta \leq_a \rho$.*

Proof. Given $\epsilon > 0$, pick open cover $(U_i)_1^r$ with $\max_i \operatorname{diam} U_i < \epsilon$. Then $(\delta, \delta_{\mathbb{N}}|^{[r]})$-computability of R and $(\delta_{\mathbb{N}}|^{[r]}, \rho)$-computability of S implies (δ, ρ)-computability of T_ϵ. $\square$

Some generalisation is possible. Consider e.g.

Lemma 2. *If (X, d) a totally bounded metric space and $\nu \in \mathcal{I}_d$ then for any $r \in \mathbb{Q}^+$ there exists finite $A \subseteq \operatorname{dom} \nu$ with $X = \cup_{a \in A} B_d(\nu(a); r)$.*

For $\nu_0, \nu_1, \nu, \lambda \in \mathcal{I}_{\mathsf{d}}$, write

$$\nu \le_{\mathsf{a}} \lambda :\iff (\forall \epsilon > 0)(\exists h \in P^{(1)})(\forall c)\Big(c \in \operatorname{dom} \nu \implies c \in (\lambda \circ h)^{-1} B(\nu(c); \epsilon)\Big).$$

$$\operatorname{dom}(\nu_0 \oplus \nu_1) := \dot{\bigcup_i}(2 \operatorname{dom} \nu_i + i), \quad \nu_0 \oplus \nu_1(2a + i) := \nu_i(a)$$

$\nu \in \mathcal{I}_{\mathsf{d}}$ **compact** if any finite open cover $(U_i)_1^r$ admits some $f \in P^{(1)}$ with

$$\operatorname{dom} \nu \subseteq f^{-1}[r] \wedge (\forall a \in \operatorname{dom} \nu) \big(\nu(a) \in U_{f(a)}\big) \qquad (1)$$

Proposition 3. *Let X be a separable metric space, ρ_ν the Cauchy representation for $\nu \in \mathcal{I}_d$. For any $\nu, \lambda \in \mathcal{I}_d$ and $\delta, \rho \in \mathcal{R}_X$,*

1. *$\nu \le_a \lambda \iff \rho_\nu \le_a \rho_\lambda$*
2. *$\delta \le \rho \implies \delta \le_a \rho$*
3. *$\delta \sqcup \rho$ is a least upper bound of $\{\delta, \rho\}$ w.r.t. $\le_a$*
4. *$\nu \oplus \lambda$ is a least upper bound of $\{\nu, \lambda\}$ w.r.t. $\le_a$*

Proof of (3): First apply (2) in $\delta_i \le \delta_0 \sqcup \delta_1$ ($i < 2$). If also $\delta_i \le_{\mathsf{a}} \rho$, say via F_i at precision ϵ ($i < 2$), then $\delta_0 \sqcup \delta_1 \le_{\mathsf{a}} \rho$ at precision ϵ via $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}, i.p \mapsto F_i(p)$. $\square$

Each $\leq_{\mathsf{a}}$ also reflexive and transitive. Thus above operations give rise to upper semilattice structures, on $\mathcal{R}_X/{\equiv_{\mathsf{a}}}$, $I_{\mathsf{a}} := \mathcal{I}_{\mathsf{d}}/{\equiv_{\mathsf{a}}}$ and $(\mathcal{I}_{\mathsf{d}} \cap X^{\mathbb{N}})/{\equiv_{\mathsf{a}}}$. If X compact, analogue of Lemma 1 for dense partial sequences implies compact $\nu \in \mathcal{I}_{\mathsf{d}}$ form a least element of I_{a} (later we show there exists compact $\nu \in \mathcal{I}_{\mathsf{d}}$).

Proposition 4. *1.* $\rho_{\leq} \not\leq_a \rho_{>}$

2. $\rho \leq_a \rho_{Cf}$

3. $\rho_{<} \leq_a \rho_{\leq}$ *and symmetrically (replace* $<, \leq$ *by* $>, \geq$*).*

Hence $\rho \leq_a \rho_{Cf}, \rho_{\leq}, \rho_{\geq}$ *and* $\rho_{b,n} \leq_a \rho_{Cf}, \rho_{\geq}, \rho_{\leq}$ *and (3) are the only* $\leq_a$*-reductions not shown in the left figure.*

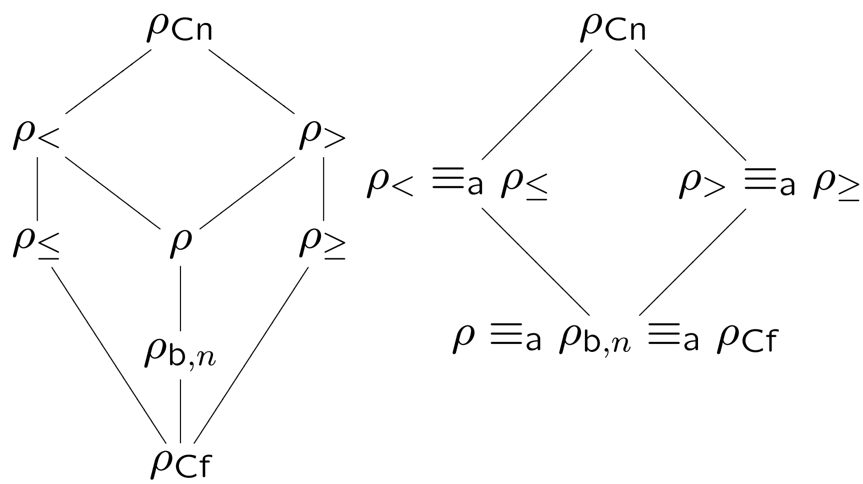
Proof. (1): Let F realise $\rho_{\leq} \leq_{\mathsf{a}} \rho_{>}$ to precision ϵ; where $p \in \rho_{\leq}^{-1}\{x\}$ enumerates all rationals $\leq x$, $q = F(p)$ enumerates strict right Dedekind cut of some y s.t. $|x-y| < \epsilon$. q_0 is output after finitely many steps, with only finite prefix p^N of p read from input. Let $p' \in \rho_{\leq}^{-1}\{z\}$ for some $z \geq \nu_{\mathbb{Q}}(q_0) + \epsilon$ $(> y + \epsilon > x)$ with $(p')^N = p^N$. Then $F(p')_0 = q_0$, so $(\rho_{<} \circ F)(p') < \nu_{\mathbb{Q}}(q_0) \leq z - \epsilon$, contradiction.

(3): Recall the $\mathbf{T}_0$-topology $\tau_{<} = \{(x,\infty) \mid x \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$ on $\mathbb{R}$ (with respect to which $\rho_{<}$ is admissible) and the following

Lemma 5. *For any* $D \subseteq \mathbb{R}$*, a function* $f :\subseteq \mathbb{R} \to \mathbb{R}$ *with* $\operatorname{dom} f = D$ *is* $(\tau_{<}, \tau_{<})$*-continuous iff it is left-continuous and nondecreasing.*

We specify f which is $(\rho_<, \rho_\leq)$-computable and lies in the open ϵ-envelope of $\mathrm{id}_\mathbb{R}$. First, consider $f : \mathbb{R} \to \mathbb{R}$ as in Lemma 5 which is also piecewise constant: take $f(t) = c_i$ for $t_i < t \leq t_{i+1}$ where strictly increasing $(c_i)_{i\in\mathbb{Z}}, (t_i)_{i\in\mathbb{Z}}$ have $\inf t_i = -\infty$, $\sup t_i = \infty$, $c_i - \epsilon \leq t_i \wedge t_{i+1} < c_i + \epsilon$ $(i \in \mathbb{Z})$. If $(t_i)_i$ are uniformly right-computable and $(c_i)_i$ uniformly left-computable then f is $(\rho_<, \rho_<)$-computable. If $(c_i)_{i\in\mathbb{Z}} \subseteq \mathbb{R} \setminus \mathbb{Q}$ then any $(\rho_<, \rho_<)$-realiser of f also $(\rho_<, \rho_\leq)$-realises it. For instance, if $\epsilon = 2^{-j+\frac{1}{2}}$ we can take $t_n := n\epsilon$, $c_n := t_n + \frac{\epsilon}{2} = \frac{2n+1}{2}\epsilon$ $(\notin \mathbb{Q})$, $n \in \mathbb{Z}$.

Note now that (1) (with transitivity of $\leq_\mathsf{a}$) implies $\delta_0 \not\leq_\mathsf{a} \delta_1$ for all $(\delta_0, \delta_1) \in \{\rho_\leq, \rho_<, \rho_{\mathsf{Cn}}\} \times \{\rho_{\mathsf{Cf}}, \rho_{\mathsf{b},n}, \rho_\geq, \rho, \rho_>\}$. With symmetric version (exchanging $<, \leq$ with $>, \geq$), shows $\rho_{\mathsf{Cn}} \not\leq_\mathsf{a} \delta$ for all $\delta \in S \setminus \{\rho_{\mathsf{Cn}}\}$, where $S := \{\rho_{\mathsf{Cf}}, \rho_\leq, \rho_\geq, \rho_{\mathsf{b},n}, \rho, \rho_<, \rho_>, \rho_{\mathsf{Cn}}\}$ (for convenience we fix n), while $\delta \leq \rho_{\mathsf{Cn}}$ for all $\delta \in S$. For $\delta_0 \in \{\rho_\leq, \rho_\geq\}$ again (1) and figure completely determine either $\delta_0 \not\leq_\mathsf{a} \delta_1$ or $\delta_0 \leq \delta_1$ as $\delta_1 \in S$ varies. Case $\delta_0 \in \{\rho_<, \rho_>\}$ is similar except where (3) and its symmetric version apply. So assume $\delta_0 \in \{\rho_{\mathsf{Cf}}, \rho_{\mathsf{b},n}, \rho\}$, and $\delta_1 \in S \setminus \{\rho_<, \rho_>, \rho_{\mathsf{Cn}}\}$ (otherwise $\delta_0 \leq \delta_1$). By (2) we already have $\delta_0 \leq_\mathsf{a} \delta_1$, which completes the proof. $\square$



From the figure ($\rho \equiv_{\mathsf{a}} \rho_{\mathsf{b},n} \equiv_{\mathsf{a}} \rho_{\mathsf{Cf}}$) and known facts it follows $\le_{\mathsf{a}}$ does not imply continuous reducibility $\le_{\mathsf{t}}$ (nor does $\equiv_{\mathsf{a}}$ guarantee the same final topology). The converse inclusion $(\le_{\mathsf{t}}) \subseteq (\le_{\mathsf{a}})$ also fails in general, using next construction from [**?**] (with Proposition 3(1)).

Definition 6. Let (Y, ν) be a numbered set with $|Y| \ge 2$ and fix $x, y \in Y$ with $x \ne y$. For each $A \subseteq \mathbb{N}$ write

$$\nu^A(2k+i) = \nu^A_{x,y}(2k+i) := \begin{cases} \nu(k), & \text{if } i = 0 \wedge k \in \operatorname{dom} \nu, \\ x, & \text{if } i = 1 \wedge k \in A, \\ y, & \text{if } i = 1 \wedge k \in \mathbb{N} \setminus A \end{cases}$$

$(i < 2)$.

Iterating, we can compare towers constructed this way. First consider a generalisation of compact $\nu \in \mathcal{I}_{\mathsf{d}}$.

Definition 7. Let X be a separable metric space. $\nu \in \mathcal{I}_{\mathsf{d}}$ **separating at finite precision** if for any distinct $x, y \in X$ there exist open cover $(U_i)_1^r$, $V, W \in \mathcal{T}_X$, $h \in P^{(1)}$ with

$$x \in V \wedge y \in W \wedge (\forall c \in \nu^{-1}(V \cup W))(c \in h^{-1}[r] \wedge \nu(c) \in U_{h(c)}) \wedge \{i \,\big|\, U_i \cap V \neq \emptyset \neq U_i \cap W\} = \emptyset.$$

Theorem 8. *Let X be a separable metric space, $\nu_0, \ldots, \nu_n, \lambda, \lambda' \in \mathcal{I}_d$, $A_i, B \subseteq \mathbb{N}$ $(i < n \in \mathbb{N})$, $E_n := \{x_i, y_i \,\big|\, i < n\} \subseteq X$ and $x, y \in X$ distinct. Suppose ν_0 separating a.f.p., $\nu_{i+1} = (\nu_i)_{x_i, y_i}^{A_i}$ for all $i < n$ and $\lambda' = \lambda_{x,y}^B$. If $E_n \cap \{x, y\} = \emptyset$ and B is nonrecursive then $\lambda' \not\leq_{\mathsf{a}} \nu_n$.*

Proof. Fix $(U_i)_1^r, V, W, h$ as in definition of separation a.f.p., and pick $\epsilon > 0$ suff. small that $B(x;\epsilon) \subseteq V \wedge B(y;\epsilon) \subseteq W \wedge N_\epsilon(E_n) \cap \{x, y\} = \emptyset$. Also suppose $\lambda'|_{2\mathbb{N}+1} \leq_{\mathsf{a}} \nu_n$ at precision ϵ via $f \in P^{(1)}$. Write $k \in \mathbb{N}$, $a = f(2k+1) = \sum_{i=0}^n a_i 2^i$ where $a_n \in \mathbb{N}$, $(a_i)_{i<n} \subseteq \{0, 1\}$. The last requirement on ϵ means $\lambda'(2k+1) \in \{x, y\} \implies \nu_n(a) \in \operatorname{im} \nu_0 \setminus E_n$ and $a_i = 0$ for $i < n$ (inductively for $m = n, \ldots, 1$, use

$$\operatorname{im} \nu_0 \setminus E_n \ni \nu_m\Big(\sum_{j=0}^m b_j 2^j\Big) = \nu_{m-1}^{A_{m-1}}\Big(\sum_{j=1}^m b_j 2^j\Big) = \nu_{m-1}\Big(\sum_{j<m} b_{j+1} 2^j\Big)$$

where $(b_j)_0^m = (a_i)_0^n, (a_{i+1})_0^{n-1}, \ldots, (a_{i+n-1})_0^1)$. So, we get $\nu_n(a) = \nu_n(a_n 2^n) = \cdots = \nu_1(a_n 2^1) = \nu_0(a_n)$ while $g : k \mapsto a_n = \left\lfloor 2^{-n} f(2k+1) \right\rfloor$ is computable with $\operatorname{im} g \subseteq \operatorname{dom} \nu_0$.

$(\lambda'(2k+1) = x \implies g(k) \in h^{-1}B_0) \wedge (\lambda'(2k+1) = y \implies g(k) \in h^{-1}B_1)$ for all $k \in \mathbb{N}$ where $B_0 := \{i \mid B(x;\epsilon) \cap U_i \neq \emptyset\}$ and $B_1 := \{i \mid B(y;\epsilon) \cap U_i \neq \emptyset\}$. In particular, $B \leq_{\mathrm{m}} B_0$ via $h \circ g$, which implies B recursive, a contradiction. So, $\lambda' \not\leq_{\mathrm{a}} \nu_n$. $\square$

Lemma 9. *1. $A \leq_m B \implies \nu^A \leq \nu^B$,*

2. If $x \neq y \wedge \{x_i, y_i \mid i < n\} \cap \{x, y\} = \emptyset \wedge \emptyset \neq A \neq \mathbb{N} \wedge (\forall i < n)(\nu_{i+1} = (\nu_i)_{x_i,y_i}^{A_i})$ and $\lambda_{x,y}^B \leq_a (\nu_n)_{x,y}^A$ where ν_0 separating a.f.p. then $B \leq_m A$.

Proof. (1): Fix $f \in R^{(1)}$ such that $A = f^{-1}B$ and let $g : \mathbb{N} \to \mathbb{N}, 2k+i \mapsto \begin{cases} 2k, & \text{if } i = 0, \\ 2f(k)+1, & \text{if } i = 1 \end{cases}$. Then one checks $\nu^A \leq \nu^B$ via g.

(2): Fix $(U_i)_1^r, V, W, h$ as in definition of separation a.f.p. (for $x \neq y$), $\epsilon > 0$ such that $B(x;\epsilon) \subseteq$

$V \wedge B(y;\epsilon) \subseteq W \wedge N_\epsilon(E_n) \cap \{x,y\} = \emptyset$ where $E_n := \{x_i, y_i \,\big|\, i < n\}$. Let $g \in P^{(1)}$ witness $\lambda^B \leq_{\mathsf{a}} (\nu_n)^A$ at precision ϵ and denote $l : \mathbb{N} \to \mathbb{N}, k \mapsto \left\lfloor \frac{1}{2} g(2k+1) \right\rfloor$, $C_i := \{k \in \mathbb{N} \,\big|\, g(2k+1) \equiv i \pmod 2\}$ $(i = 0,1)$. Also let $B_0 := \{i \,\big|\, B(x;\epsilon) \cap U_i \neq \emptyset\}$, $B_1 := \{i \,\big|\, B(y;\epsilon) \cap U_i \neq \emptyset\}$, and choose $c \in A$, $d \in \mathbb{N} \setminus A$, $m : k \mapsto \left\lfloor 2^{-n} l(k) \right\rfloor$ and $f : \mathbb{N} \to \mathbb{N}, k \mapsto \begin{cases} c, & \text{if } k \in C_0 \wedge m(k) \in h^{-1}B_0, \\ d, & \text{if } k \in C_0 \wedge m(k) \in h^{-1}B_1, ; \\ l(k), & \text{if } k \in C_1 \end{cases}$ plainly C_0, C_1 are disjoint recursive sets (with union $\mathbb{N}$), so $f \in P^{(1)}$. We show $f \in R^{(1)}$ with $(\forall k)(k \in B \iff f(k) \in A)$.

Firstly, if $k \in C_0$ then

$$\begin{aligned} &\epsilon > d(\lambda^B(2k+1), (\nu_n \circ l)(k)) \\ &\implies (\nu_n \circ l)(k) \in \operatorname{im} \nu_0 \setminus E_n \\ &\implies l(k) = a_n 2^n \end{aligned}$$

where $a_n = m(k)$. So $(\nu_n \circ l)(k) = \nu_n(a_n.2^n) = \cdots = \nu_0(a_n.2^0) = (\nu_0 \circ m)(k)$ (this also shows $C_0 \subseteq m^{-1} \operatorname{dom} \nu_0$). Now definition of B_0, B_1

implies $(\lambda^B(2k+1)=x \implies m(k)\in h^{-1}B_0)\wedge(\lambda^B(2k+1)=y \implies m(k)\in h^{-1}B_1)$, so $k\in C_0$ implies $k\in B \iff f(k)\in A$. For $k\in C_1$, instead $\lambda^B_{x,y}(2k+1)=((\nu_n)^A_{x,y}\circ g)(2k+1)$ with $k\in B \iff l(k)\in A$ (this uses $d(x,y)\geq\epsilon$). So f has the properties required. $\square$

In particular, for any ν_n, x, y as above, the map $\alpha : A \mapsto (\nu_n)^A_{x,y}$ induces an embedding of $\leq_{\mathsf{m}}$-degrees in $\leq_{\mathsf{a}}$-degrees with least element $[\nu_n]_{\equiv_{\mathsf{a}}}$.

Randomness and density

Definition 10. Consider a bounded effective metric space (X,d,ν_0). A **denseness test** (A,a,j) has $A\subseteq\mathbb{N}$ infinite c.e., $a\in\operatorname{dom}\nu_0$, $j\in\mathbb{N}$; $\xi\in X^{\mathbb{N}}$ *fails* (A,a,j) if $\xi\in\cap_{l\in A}X^{\mathbb{N}}\setminus\sigma^{-l}\left(\alpha\langle a,j\rangle\times X^{\mathbb{N}}\right)$, *passes* (A,a,j) if $\xi\in\cup_{l\in A}\sigma^{-l}\left(\alpha\langle a,j\rangle\times X^{\mathbb{N}}\right)$; these define resp. closed, open sets in $X^{\mathbb{N}}$.

$\mathcal{I}^{\mathsf{rand}}_{\mathsf{d}} := \{\xi\in X^{\mathbb{N}} \mid \xi \text{ passes all denseness tests}\}.$

We introduce another generalisation of the definition of compact sequences.

Definition 11. Suppose X a separable metric space, $(U_i)_1^r$ a finite open cover, $\epsilon > 0$ with each $N_\epsilon(U_i)$ nondense and $\nu \in \mathcal{I}_{\mathrm{d}}$, $f \in P^{(1)}$ s.t. (1) holds. Then ν is *properly covering*.

If $\nu \in \mathcal{I}_{\mathrm{d}}$ is properly covering then $|X| \geq 2$; any separating a.f.p. dense sequence in a compact space X with $|X| \geq 2$ is properly covering.

Proof. For each $x \in X$ we have $(U_i(x))_1^{r_x}$, $\mathcal{T}_X \ni V_x \ni x$ and $h_x \in P^{(1)}$ s.t. $(\forall c \in \nu^{-1}V_x)(\nu(c) \in U_{h_x(c)}(x))$. By compactness there exist s and $(x_i)_{i<s} \subseteq X$ with $X = \cup_{i<s} V_{x_i}$. We can take a formal disjoint union of $(U_j(x_i))_{j \in [r_{x_i}]}$ $(i < s)$, say $(U_i)_{i \in [r]}$ where $r := \sum_{i<s} r_{x_i}$, and by adding appropriate constants to each h_{x_i} $(i < s)$ we get $h \in P^{(1)}$ with $(\forall c \in \operatorname{dom} \nu)(\nu(c) \in U_{h(c)})$. $\square$

Proper covering does not imply separation at finite precision.

Proposition 12. *1. Suppose $\nu_0, \ldots, \nu_n \in \mathcal{I}_{\mathsf{d}}$, $(\forall i < n)(\nu_{i+1} = (\nu_i)^{A_i}_{x_i,y_i})$, ν_0 total & properly covering, and $\lambda \in \mathcal{I}^{\mathsf{rand}}_{\mathsf{d}}$. Then $\lambda \not\leq_{\mathsf{a}} \nu_n$.*

2. Suppose $|X| \geq 2$. For any $\nu \in \mathcal{I}^{\mathsf{rand}}_{\mathsf{d}}$ there exists $\lambda \in X^{\mathbb{N}} \setminus \mathcal{I}^{\mathsf{rand}}_{\mathsf{d}}$ with $\nu \leq \lambda$.

3. $\mathcal{I}^{\mathsf{rand}}_{\mathsf{d}}$ is closed under $\oplus$.

Proof. (1): Suppose $\lambda \leq_{\mathsf{a}} \nu_n$ via $f \in P^{(1)}$ at precision ϵ, where ϵ suff. small that $N_\epsilon(\{x_i, y_i\})$ and $N_\epsilon(U_j)$ nondense for each $i < n$ and each $j \in [r]$, for some open cover $(U_j)^r_1$. We have $\operatorname{dom} \nu_n = \mathbb{N} = \dot{\bigcup}_{i<n} A_i \,\dot{\cup}\, 2^n\mathbb{N}$ where each A_i is infinite c.e. with $\nu_n(A_i) = \{x_i, y_i\} \subseteq \operatorname{im} \nu_{i+1}$ $(i < n)$. Since λ total, $f \in R^{(1)}$ with some $A_i \cap \operatorname{im} f$ $(i < n)$ or $2^n\mathbb{N} \cap \operatorname{im} f$ infinite (by pigeonhole principle). Each of these sets is c.e. If $A_i \cap \operatorname{im} f$ infinite, its ν_n-image ($\subseteq \{x_i, y_i\}$) lies within ϵ of $\lambda(f^{-1}A_i)$ which is dense, contradicting choice of ϵ. If $2^n\mathbb{N} \cap \operatorname{im} f$ infinite, let h be as in definition of ν_0 for cover $(U_i)^r_1$. Then each $h^{-1}\{i\}$ is c.e., so $f^{-1}(2^n(h^{-1}\{i\}))$ is c.e., and at least one such set is infinite, so has λ-image dense & $\subseteq N_\epsilon(U_i)$ (contradicting choice of $(U_i)_1^r, h$).

(2): For any $\nu \in \mathcal{I}_{\mathrm{d}}^{\mathrm{rand}} \not\ni \lambda$ we have $\nu \oplus \lambda \notin \mathcal{I}_{\mathrm{d}}^{\mathrm{rand}}$. Such λ can always be found e.g. as a nondense total sequence (extend $\oplus$ definition).

(3): Let $\lambda_0, \lambda_1 \in \mathcal{I}_{\mathrm{d}}^{\mathrm{rand}}$. For any denseness test (A, a, j) we have some $i < 2$ such that $A \cap (2\mathbb{N} + i)$ is infinite. Clearly $\lambda_i \leq \lambda_0 \oplus \lambda_1$ via (injective total recursive) $h : k \mapsto 2k + i$, and $B := h^{-1}(A \cap (2\mathbb{N} + i))$ is infinite c.e. We know λ_i passes the denseness test (B, a, j), say $k \in B \cap \lambda_i^{-1}\alpha\langle a, j\rangle$, so $(\lambda_0 \oplus \lambda_1)(h(k)) = \lambda_i(k)$ shows $\lambda_0 \oplus \lambda_1$ passes (A, a, j). $\square$

Remains to establish $\mathcal{I}_{\mathsf{d}}^{\mathsf{rand}} \neq \emptyset$. For topological space Y, a continuous surjection $T : Y \to Y$ is *one-sided topologically mixing* if

$$(\forall U, V \in \mathcal{T}_Y \setminus \{\emptyset\})(\exists N \in \mathbb{N})(\forall n)\,(n \geq N \implies T^n(U) \cap V \neq \emptyset)\,.$$

One checks

Lemma 13. *For any separable metrizable X and $Y = X^{\mathbb{N}}$, left shift $\sigma : Y \to Y$ is one-sided topologically mixing (w.r.t. product topology).*

More generally, consider a complete effective metric space (Y, d, ν) with ideal ball numbering α and continuous one-sided top. mixing $T : Y \to Y$. By definition, each $\cup_{m \in A} T^{-m} V$ dense ($V \in \mathcal{T}_Y \setminus \{\emptyset\}$, A infinite), so in particular $R_A := \cap_{a \in \mathrm{dom}\,\alpha} \cup_{m \in A} T^{-m} \alpha(a)$ and $R := \cap\{R_A \,\big|\, A \text{ infinite c.e.}\}$ dense $\mathcal{G}_\delta$ in Y (by Baire category theorem). We now apply to $Y = X^{\mathbb{N}}$. Recall a formal inclusion $\sqsubset$ of total basis numberings α, β of space X has the *weak basis property* if $(\forall b)(\forall x \in X)(\exists a)(x \in \beta(b) \implies x \in \alpha(a) \wedge a \sqsubset b)$.

Proposition 14. *Let (X, d, ν_0) be a complete bounded effective metric space with ν_0 total, and equip $X^{\mathbb{N}}$ with the (bounded) product metric $\hat{d}(\xi,\eta) := \sum_{i\in\mathbb{N}} 2^{-i-1} d(\xi_i,\eta_i)$ and dense sequence $\gamma : \mathbb{N} \to X^{\mathbb{N}}$ defined by $\gamma(\langle w\rangle)(i) := \begin{cases} \nu_0(w_i), & \text{if } i < |w|, \\ \nu_0(i), & \text{if } i \geq |w| \end{cases}$. Then basis numberings defined by*

$$\alpha_{X^{\mathbb{N}}} : \mathbb{N} \to \mathcal{T}_{X^{\mathbb{N}}}, \langle a, r\rangle \mapsto B_{\hat{d}}(\gamma(a); \nu_{\mathbb{Q}^+}(r)),$$
$$\beta : \mathbb{N} \to \mathcal{T}_{X^{\mathbb{N}}}, \langle a, r, N\rangle \mapsto \prod_{i<N} B_d(\gamma(a)_i; \nu_{\mathbb{Q}^+}(r)) \times X^{\mathbb{N}}$$

are effectively equivalent, and any $\xi \in X^{\mathbb{N}}$ has

$$\Big((\sigma^l \xi)_{l\in\mathbb{N}} \in \mathcal{I}_d^{\mathit{rand}}|^{X^{\mathbb{N}}} \iff \xi \in R|^{X^{\mathbb{N}},\sigma} \implies \xi \in \mathcal{I}_d^{\mathit{rand}}|^{X}\Big);$$

in particular $\mathcal{I}_d^{\mathit{rand}}|^{X}$ is residual in $X^{\mathbb{N}}$.

Proof. It is convenient to use the following binary relations on $\mathbb{N}$:

$$\langle a, r\rangle \sqsubset \langle b, q, N\rangle :\iff \max_{i<N} d(\gamma(a)_i, \gamma(b)_i) + 2^{i+1}\nu_{\mathbb{Q}^+}(r) < \nu_{\mathbb{Q}^+}(q),$$

$$\langle a, r, N\rangle \sqsubset' \langle b, q\rangle :\iff \sum_{i<N} 2^{-i-1}(\nu_{\mathbb{Q}^+}(r) + d(\gamma(a)_i, \gamma(b)_i)) + C_0.2^{-N} < \nu_{\mathbb{Q}^+}(q);$$

here $\mathbb{Q}^+ \ni C_0 \geq \mathrm{diam}(X,d)$ is a fixed bound. We first check $\sqsubset$, $\sqsubset'$ are c.e. formal inclusions (of $\alpha_{X^{\mathbb{N}}}$ in β, resp. of β in $\alpha_{X^{\mathbb{N}}}$), each with the weak basis property.

Next denote $B_{\langle a,r\rangle} := \{\langle b,q\rangle \,\big|\, \langle b,q\rangle \sqsubset \langle\langle a\rangle, r, 1\rangle\}$ ($a, r \in \mathbb{N}$). Clearly $B_{\langle a,r\rangle}$ is nonempty and c.e. uniformly in $a, r \in \mathbb{N}$, and for any $A \subseteq \mathbb{N}$ the condition $(\forall a,r)\left(\xi \in \cup_{l\in A}\sigma^{-l}(\alpha\langle a,r\rangle \times X^{\mathbb{N}})\right)$ is equivalent to $(\forall a,r)(\exists\langle b,q\rangle \in B_{\langle a,r\rangle})\left(\xi \in \cup_{l\in A}\sigma^{-l}\alpha_{X^{\mathbb{N}}}\langle b,q\rangle\right)$. In particular, this is implied by $(\forall b,q)\left(\xi \in \cup_{l\in A}\sigma^{-l}\alpha_{X^{\mathbb{N}}}\langle b,q\rangle\right)$, or equivalently $(\forall b,q)(\exists k \in A)\left((\sigma^{l+k}\xi)_{l\in\mathbb{N}} \in \alpha_{X^{\mathbb{N}}}\langle b,q\rangle \times (X^{\mathbb{N}})^{\mathbb{N}}\right)$. Quantifying over infinite c.e. A, thus ($\xi \in R|^{X^{\mathbb{N}},\sigma} \iff$) $(\sigma^l\xi)_{l\in\mathbb{N}} \in \mathcal{I}_{\mathrm{d}}^{\mathrm{rand}}|^{X^{\mathbb{N}}}$ implies $\xi \in \mathcal{I}_{\mathrm{d}}^{\mathrm{rand}}|^{X}$. $\square$

Example: $X = \{0,1\}$ corresponds (easily checked) to $\mathcal{I}_{\mathrm{d}}^{\mathrm{rand}}|^{X} = \{\chi_B \,\big|\, B \subseteq \mathbb{N} \text{ bi-immune}\}$, i.e. those sets B for which neither B nor $\mathbb{N} \setminus B$ contains an infinite c.e. set.

Comparison with other tests

For any $A \subseteq \mathbb{N}$, $N \in \Pi^0_1(X)$ denote

$$F_{A,N} := \prod_{i \in \mathbb{N}} W_i \text{ where } W_i = \begin{cases} X, & \text{if } i \notin A, \\ N, & \text{if } i \in A \end{cases}.$$

$F_{A,X\setminus\alpha\langle a,j\rangle} = \{\xi \,\big|\, \xi \text{ fails } (A,a,j)\}$ for denseness test (A,a,j),

$X^{\mathbb{N}} \setminus \mathcal{I}^{\mathrm{rand}}_{\mathrm{d}} = \cup\{F_{A,N} \,\big|\, A \text{ infinite c.e., } N = X\setminus\alpha\langle a,j\rangle, a,j \in \mathbb{N}\}$.

??For X effectively compact,

$$f : \{0,1\}^{\mathbb{N}} \times \mathcal{K}_>(X) \to \mathcal{K}_>(X^{\mathbb{N}}), (\chi_A, K) \mapsto F_{A,K}$$

computable. More generally, suppose $|X| \geq 2$. If μ_0 is a measure positive on nonempty open sets, for any test (A,a,j) we have $\mu_0(X \setminus B_0) < 1$ for $B_0 := \alpha\langle a,j\rangle$, so $F_{A,X\setminus B_0}$ is a closed λ-nullset. Here λ is the product measure on $X^{\mathbb{N}}$ corresponding to measure μ_0 on X; also one can show λ a computable probability measure if μ is and X bounded complete.

Effectiveness and approximability

When choosing naming system γ for a given space Y, several types of requirements [**?**, §2.7]:

(a) require $f_i :\subseteq X_i \to Y$ computable $(i \in I)$

(b) require $f_i :\subseteq Y \to Z_i$ computable $(i \in I)$

(c) require operations $f_i :\subseteq Y^{n_i} \to Y$ computable $(i \in I)$

For example, types (a)+(c) and (b) resp. addressed by **Proposition 15.** *Fix represented sets* (X_i, ρ_i)*, maps* $f_i :\subseteq X_i \to Y$*,* $g_i :\subseteq Y \to Y$ $(i \in \mathbb{N})$ *& suppose*

$Y = \cup_{i\in\mathbb{N},w\in\mathbb{N}^*}\widehat{g}_w(\mathrm{im}\, f_i \cap \mathrm{dom}\, \widehat{g}_w)$ *where* $\widehat{g}_w := g_{w_{|w|-1}} \circ \ldots \circ g_{w_0}$.

$\delta :\subseteq \mathbb{N}^{\mathbb{N}} \to Y$ *defined by* $\mathrm{dom}\,\delta = \dot\cup_{i\in\mathbb{N},w\in\mathbb{N}^*} i.\langle w\rangle.(\rho_i^{-1} f_i^{-1}\,\mathrm{dom}\,\widehat{g}_w)$ *and* $\delta(i.\langle w\rangle.p) := (\widehat{g}_w \circ f_i \circ \rho_i)(p)$ *has* $\delta \in \mathcal{R}_Y$*.* f_i *is* (ρ_i, δ)*-computable and* g_i *is* (δ, δ)*-computable* $(i \in \mathbb{N})$*.*

Proposition 16. *Fix represented sets* (Y_i, δ_i)*,* $f_i :\subseteq X \to Y_i$ $(i \in \mathbb{N})$ *where* $|X| \leq 2^{\aleph_0}$*. Then there exists* $\rho \in \mathcal{R}_X$ *such that each* f_i *is strongly* (ρ, δ_i)*-computable* $(i \in \mathbb{N})$*.*

More sophisticated results for type (c), see [**?**, Thm 2.7.15], [**?**], [**?**].

For requirements of type (a)+(b) or (b)+(c), in general there exist counterexamples:

Proposition 17. *1. there exist sets X, Y, Z, maps $f :\subseteq X \to Y$, $g :\subseteq Y \to Z$ and total numberings ν, μ of X, Z respectively, such that $\neg(\exists \lambda :\subseteq \mathbb{N} \to Y)(f \circ \nu \leq \lambda \wedge g \circ \lambda \leq \mu)$.*

2. there exist sets Y, Z, maps $f :\subseteq Y \to Y$, $g :\subseteq Y \to Z$ and total numbering μ of Z such that $\neg(\exists \lambda \in \mathsf{TN}(Y))(f \circ \lambda \leq \lambda \wedge g \circ \lambda \leq \mu)$.

3. there exist separable metric spaces Y, Z, maps $f :\subseteq Y \to Y$, $g :\subseteq Y \to Z$ and $\mu \in \mathcal{I}_d^Z \cap Z^{\mathbb{N}}$ such that $\neg(\exists \lambda \in \mathcal{I}_d^Y)(f \circ \lambda \leq \lambda \wedge g \circ \lambda \leq \mu)$.

Proof. (1): $X = Y = Z$ denumerable, $f = g = \mathrm{id}_X$, ν total injective numbering of X, μ a total numbering s.t. $\nu \not\leq \mu$ (e.g. an incomparable total injective numbering, of which there are $2^{\aleph_0}$; see [**?**]).
(2): $Y = Z$ denumerable, μ a total injective numbering, $h : \mathbb{N} \to \mathbb{N}$ a nonrecursive bijection with $N_h : \mathbb{N} \to \bar{\mathbb{N}}, k \mapsto \#\,\mathrm{Fix}(h^k)$ not lower semicomputable, $f = \mu \circ h \circ \mu^{-1}$, $g = \mathrm{id}_Y$. If $\lambda \in \mathsf{TN}(Y)$ and $n \in P^{(1)}$ (λ, μ)-realises g, $=_Y$ is $[\lambda, \lambda]$-decidable via $\langle a, b\rangle \mapsto \delta_{n(a),n(b)}$ (since g, μ injective). So $N_{f,M} : \mathbb{N} \to \mathbb{N}, k \mapsto \#\,\mathrm{Fix}(f^k|_{\lambda[0,M)})$ $(M \geq 1)$ are uniformly effective provided also f is (λ, λ)-effective: consider

$$a_n := \mu a < M\left((f^k \circ \lambda)(a) = \lambda(a) \notin \{\lambda(a_i) \,\middle|\, i < n\}\right)$$

with convention $\mu a < M\,(\mathsf{False}) = M$, inductively for $n \leq M$ and $N := \mu n\,(a_n = M)$ $(\leq M)$. Then $\lambda|_{\{a_i | i<N\}}$ is injective with image $\mathsf{Fix}(f^k|_{\lambda[0,M)})$, so $N_{f,M}(k) = N$.

Further, $(f^k \circ \lambda)(a) = \lambda(a)$ iff $(h^k \circ \mu^{-1} \circ \lambda)(a) = (\mu^{-1} \circ \lambda)(a)$, so $\mathsf{Fix}(h^k|_{(\mu^{-1}\circ\lambda)[0,M)}) = \mu^{-1}\,\mathsf{Fix}(f^k|_{\lambda[0,M)})$. In unions over increasing M, $\mathsf{Fix}(h^k) = \mu^{-1}\,\mathsf{Fix}(f^k)$ and $N_h(k) = \sup_M N_{f,M}(k)$. This contradicts choice of h.
(3): $Y = Z = \mathbb{R}$, $f = \mathrm{id}_{\mathbb{R}} + \alpha$ $(\alpha \in \mathbb{R} \setminus \mathbb{Q})$, $g = \mathrm{id}_{\mathbb{R}}$, $\mu = \nu_{\mathbb{Q}}$. The only solution for $\lambda :\subseteq \mathbb{N} \to Y$ is then the empty partial sequence. $\square$

Next, consider (a),(b),(c) when maps are approximable rather than computable. Here $f :\subseteq X \to Y$ *approximable* w.r.t. $\nu \in \mathcal{I}_{\mathsf{d}}^X$, $\lambda \in \mathcal{I}_{\mathsf{d}}^Y$ if $f \circ \nu \leq_{\mathsf{a}} \lambda$ (extending definition of $\leq_{\mathsf{a}}$).

If X uniformly discrete, (1), (2) transfer since $\nu \leq \lambda$ iff $\nu \leq_{\mathsf{a}} \lambda$ for any $\nu, \lambda \in \mathcal{I}_{\mathsf{d}}^X = \mathsf{TN}(X)$. If X compact, instead take $\nu_0 \in \mathcal{I}_{\mathsf{d}}^X$ compact and pick ν, μ $\leq_{\mathsf{a}}$-incomparable. Then
$\neg(\exists\lambda :\subseteq \mathbb{N} \to Y)(f \circ \nu \leq_{\mathsf{a}} \lambda \wedge g \circ \lambda \leq_{\mathsf{a}} \mu)$.

Lemma 18. *If X compact, $\nu \in \mathcal{I}_d$ compact and $\pi : X \to Y$ a continuous surjection then $\pi \circ \nu$ is compact.*

Proof. Uses Lebesgue numbers and uniform continuity of π; see [**?**]. $\square$

Corollary 19. *Let Y, Z be compact metric spaces, $f : Y \to Y$, $g : Y \to Z$ (total) continuous surjective and $\mu \in \mathcal{I}_d^Z$. For any compact $\lambda \in \mathcal{I}_d^Y$ we have $f \circ \lambda \leq_a \lambda \wedge g \circ \lambda \leq_a \mu$.*

Proof. Follows from Lemma 18 and analogue of Lemma 1. $\square$

Given existence of compact $\lambda \in \mathcal{I}_d^Y$, this is a positive result for requirements of type (b)+(c).

Requirements of type (a),(b)

For $\nu \in \mathcal{I}_X$ we denote the cylindrification by $\tilde{\nu}$: $\mathrm{dom}\,\tilde{\nu} = \langle \mathrm{dom}\,\nu, \mathbb{N} \rangle$, $\tilde{\nu}\langle i, j\rangle := \nu(i)$.

Proposition 20. *Fix separable metric spaces X, Y_i, maps $\Psi_i : X \to Y_i$ and $\lambda_i \in \mathcal{I}_{\mathsf{d}}^{Y_i}$ $(i < n)$. There exists $\nu \in \mathcal{I}_{\mathsf{d}}^{X}$ s.t. $(\forall i < n)\Psi_i \circ \nu \leq_{\mathsf{a}} \lambda_i$.*

Proof. Fix $\nu_0 \in \mathcal{I}_{\mathsf{d}} \cap X^{\mathbb{N}}$ and (for $i < n$, $k \in \mathbb{N}$)

$$a_k^{(i)} := \mu n\big(n \in \mathrm{dom}\,\tilde{\lambda}_i \wedge \neg(\exists j < k)(n = a_j) \wedge \\ d((\Psi_i \circ \nu_0)(k), \tilde{\lambda}_i(n)) < 2^{-k}\big).$$

We denote $\nu\langle a_k^{(0)}, \ldots, a_k^{(n-1)}\rangle := \nu_0(k)$, with ν undefined elsewhere. For any $\epsilon > 0$, $\max_{i<n} d((\Psi_i \circ \nu)\langle a_k^{(0)}, \ldots, a_k^{(n-1)}\rangle, \tilde{\lambda}_i(a_k^{(i)})) < \epsilon$ fails only for finitely many k (among those with $\epsilon \leq 2^{-k}$). We also have $\mathrm{im}\,\nu = \mathrm{im}\,\nu_0$ and $\tilde{\lambda}_i \equiv \lambda_i$ for each $i < n$. $\square$

Proposition 21. *Let X, Y be separable metric spaces, $\Psi : X \to Y$, $\nu \in \mathcal{I}_{\mathsf{d}}^{X}$, $\lambda \in \mathcal{I}_{\mathsf{d}}^{Y}$ as appropriate. Then:*

1. *for any ν there exists λ with $\Psi \circ \nu \leq \lambda$; if Ψ continuous onto Y, there exists λ with $\Psi \circ \nu \equiv \lambda$,*

2. *if Ψ open, for any λ there exists ν with $\Psi \circ \nu \leq \lambda$,*

3. *for any ν there exists λ with $\Psi \circ \nu \leq_{\mathsf{a}} \lambda$,*

4. *for any λ there exists ν with $\Psi \circ \nu \leq_{\mathsf{a}} \lambda$.*

Proof. (2): Each $\Psi^{-1}\{\lambda(a)\}$ separable ($a \in \operatorname{dom} \lambda$) so if nonempty fix a dense subsequence $(x_i^{(a)})_{i \in \mathbb{N}}$ and let $\nu\langle a, i\rangle := x_i^{(a)}$ ($i \in \mathbb{N}$) (ν undefined elsewhere). Since Ψ open, $\Psi^{-1} \operatorname{im} \lambda$ is dense, hence so is $\operatorname{im} \nu$. $\square$

For (4), if Ψ onto we can construct $\nu \in \mathcal{I}_{\mathsf{d}}^{X}$ in a different way. First, pick $\nu_0 \in \mathcal{I}_{\mathsf{d}}^{X} \cap X^{\mathbb{N}}$ and $\lambda \in \mathcal{I}_{\mathsf{d}}^{Y}$. Let $\operatorname{dom} \nu = \{a_k \,|\, k \in \mathbb{N}\}$, $\nu(a_k) = \nu_0(k)$ for

$$a_k := \mu n \left(n \in \operatorname{dom} \tilde{\lambda} \wedge \neg(\exists j < k)(n = a_j) \wedge \tilde{\lambda}(n) \in U_k\right) \quad (k \in \mathbb{N})$$

where $(U_k)_{k\in\mathbb{N}} \subseteq \mathcal{T}_Y \setminus \{\emptyset\}$ has the property $(\forall x \in \operatorname{im}\lambda)(\forall N)(\exists k \geq N)(U_k \ni x)$. In place of $\operatorname{dom}\nu \subseteq \operatorname{dom}\tilde{\lambda}$ we now have equality: for c_l the l^{th} element of $\operatorname{dom}\tilde{\lambda}$ in ascending order, and $N_l := \mu k \left(U_k \ni \tilde{\lambda}(c_l) \wedge (\forall l' < l)(k > N_{l'})\right)$, we show $c_l \in \{a_j \,\middle|\, j \leq N_l\}$ $(l \in \mathbb{N})$. First, $N_0 = \mu k \left(U_k \ni \tilde{\lambda}(c_0)\right)$, so $\neg(\exists j < N_0)(c_0 = a_j)$ (as $\tilde{\lambda}(a_j) \in U_j$) and then $a_{N_0} = c_0$. Secondly, $a_{N_{l+1}} = \min(\tilde{\lambda}^{-1}U_{N_{l+1}}) \setminus \{a_j \,\middle|\, j < N_{l+1}\}$ and $U_{N_{l+1}} \ni \tilde{\lambda}(c_{l+1})$ so if $c_{l+1} \notin \{a_j \,\middle|\, j \leq N_{l+1}\}$ then $a_{N_{l+1}} < c_{l+1}$, implying $(\exists k \leq l) a_{N_{l+1}} = c_k \in \{a_j \,\middle|\, j \leq N_k\}$. But then $N_{l+1} > N_l \geq N_k$ contradicts injectivity of $(a_i)_{i\in\mathbb{N}}$.

To ensure also $\Psi \circ \nu \equiv_{\mathsf{a}} \tilde{\lambda}$, we assume compactness. We also note two improvements on Proposition 21(2) with similar conditions.

Proposition 22. *Suppose X, Y are separable metric spaces, $\lambda \in \mathcal{I}_d^Y$, $\Psi : X \to Y$ any map.*

1. *If Ψ onto and Y compact, there exists $\nu \in \mathcal{I}_d^X$ such that $\Psi \circ \nu \equiv_a \lambda$,*

2. *If $\overline{\Psi^{-1}\operatorname{im}\lambda} = X$ and each $\Psi^{-1}\{y\}$ is compact ($y \in \operatorname{im}\lambda$), there exists $\nu \in \mathcal{I}_d^X$ such that $\Psi \circ \nu = \tilde{\lambda}$.*

3. *If $\overline{\Psi^{-1}\operatorname{im}\lambda} = X$ and X compact, there exists compact $\nu \in \mathcal{I}_d^X$ such that $\Psi \circ \nu \leq \lambda$.*

Proof. (1): In view of the above, it remains to construct $(U_k)_k$ appropriately. As Y compact, let ν_0 be such that $\nu_0|_{2\mathbb{N}+1}$ is dense and $2(\sum_{j<l} n_j + i) \in \nu_0^{-1}\Psi^{-1}\{y_i^l\}$ ($l \in \mathbb{N}, i < n_l$) where $\{y_i^l \mid i < n_l\}$ is a 2^{-l}-spanning set for Y. Then let $U_k := B((\Psi \circ \nu_0)(k); 2^{-b_k})$ where $b_k = l$ if $2\sum_{j<l} n_j \leq k < 2\sum_{j\leq l} n_j$. This ensures $\lim_{k\to\infty} b_k = \infty$ and so $\Psi \circ \nu \equiv_{\mathsf{a}} \tilde{\lambda}$.

(2): For each $a \in \operatorname{dom}\lambda$, $k \in \mathbb{N}$ there is a finite 2^{-k}-spanning subset of $\Psi^{-1}\{\lambda(a)\}$, say $(z_i(a,k))_{i<m(a,k)}$. Let $\nu\langle a, \sum_{j<k} m(a,j)+i\rangle := z_i(a,k)$ for each $i < m(a,k)$ (also $\nu\langle a, m\rangle\uparrow$ if $a \notin \operatorname{dom}\lambda$). Plainly $(\Psi \circ \nu)\langle a, m\rangle = \lambda(a) = \tilde{\lambda}\langle a, m\rangle$ whenever $a \in \operatorname{dom}\lambda \wedge m \in \mathbb{N}$, and both sides undefined for

$a \notin \operatorname{dom}\lambda$. Given $x \in X$, $k \in \mathbb{N}$ there exist $a \in \operatorname{dom}\lambda$, $y \in \Psi^{-1}\{\lambda(a)\} \cap B(x; 2^{-k-1})$ and $i < m(a, k+1)$ s.t. $z_i(a, k+1) \in B(y; 2^{-k-1}) \cap \operatorname{im}\nu$. Then $d_{\operatorname{im}\nu}(x) < 2^{-k-1} + 2^{-k-1}$, and k was arbitrary. $\square$

We describe a general extension of (3) (of type (b)+(c)):

Proposition 23. *Let X, Y be separable metric spaces, X compact, $\lambda \in \mathcal{I}_{\mathrm{d}}^{Y}$, $g :\subseteq X \to Y$ any map with $\overline{g^{-1}\operatorname{im}\lambda} = X$, $\mathcal{F}$ a denumerable family of (total) maps $X \to X$. Also take $(\theta_k)_{k\in\mathbb{N}} \subseteq (0, \infty)$ with $\inf_k \theta_k = 0$, $E_l \subseteq g^{-1}\operatorname{im}\lambda$ a finite θ_l-spanning set ($l \in \mathbb{N}$). Suppose $\sim_l$ ($l \in \mathbb{N}$) are equivalence relations on $\mathcal{F}$ s.t. $\mathcal{F}/{\sim_l}$ finite,*

$$(\forall y \in E_l)(\forall C \in \mathcal{F}/{\sim_l})(\forall i \leq l)(\exists z \in E_i)\{fy \,\big|\, f \in C\} \subseteq B(z; \theta_i), \tag{2}$$

and which are increasingly fine & satisfy separating condition $(\forall f, f' \in \mathcal{F})(\exists l)(f \neq f' \implies f \not\sim_l f')$. Then there exists compact $\nu \in \mathcal{I}_{\mathrm{d}}^{X}$ with $g \circ \nu \leq \lambda$ and $(\forall f \in \mathcal{F})(f \circ \nu \leq_{\mathrm{a}} \nu)$.

Proof. Write $E_k = \{y_1^k, \dots, y_{r_k}^k\} \subseteq g^{-1} \operatorname{im} \lambda$ (a θ_k-spanning set for X, $k \in \mathbb{N}$). Based on $\sim_l$ ($l \in \mathbb{N}$) and an injective total numbering $e \mapsto f_e$ of $\mathcal{F}$, we define ν with $\operatorname{im} \nu = \cup_k E_k$; surjectivity will follow from (2), density from $\inf_k \theta_k = 0$. Separating condition ensures the partitions $\mathcal{P}_l := \mathcal{F}/{\sim_l}$ ($l \in \mathbb{N}$) (with $(\forall l)\mathcal{P}_{l+1} \geq \mathcal{P}_l$) are *generating*: $|\cap_l A_l| \leq 1$ for any $A_l \in \mathcal{P}_l$ ($l \in \mathbb{N}$). If each $\mathcal{P}_l$ is endowed with a numbering $\nu_l :\subseteq \mathbb{N} \to \mathcal{P}_l$, this leads to a representation $\rho :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathcal{F}$: $\operatorname{dom} \rho = \{p \in \mathbb{N}^{\mathbb{N}} \mid (\forall l)(p_l \in \operatorname{dom} \nu_l) \wedge \cap_l \nu_l(p_l) \neq \emptyset\}$, $\rho(p) \in \cap_l \nu_l(p_l)$.

Will return to numbering $e \mapsto f_e$ and representation ρ of $\mathcal{F}$. Requirement for $f \in \mathcal{F}$ to be (ν, ν)-approximable equivalent to $f \circ \nu$ compact — (roughly speaking) suggests to store approximations to each f in each ν-name a. Form of ν will be

$$a = \langle l; j_0, \dots, j_l; j_0^{(0)}, \dots, j_l^{(0)}; \dots; j_0^{(m)}, \dots, j_l^{(m)}; \widehat{b}\rangle \in \operatorname{dom} \nu,$$

$\nu(a) = y_{j_l}^l$;

$(\forall i < l) d(y_{j_i}^i, y_{j_l}^l) < \theta_i$, $\widehat{b} = b(l, j_l)$ for a fixed choice function $b :\subseteq \mathbb{N}^2 \to \mathbb{N}$ with $\operatorname{im} b \subseteq \operatorname{dom} \lambda$, $\operatorname{dom} b =$

$\{(k,j) \mid 1 \le j \le r_k\}$ and $g(y^k_j) = (\lambda \circ b)(k,j)$ for all $(k,j) \in \operatorname{dom} b$, plus further conditions on m and $j^{(q)}_i$ $(q \le m,\ i \le l)$. Namely, assuming $\operatorname{dom}\nu_l$ finite, want $m \ge \max(\operatorname{dom}\nu_l)$ and

$$d(fy^l_{j_l}, y^i_{j_i^{(p_l)}}) < \theta_i \text{ whenever } i \le l \in \mathbb{N}, f \in \mathcal{F}, p \in \rho^{-1}\{f\}.$$

One checks existence of suitable $j^{(q)}_i$ $(q \in \operatorname{dom}\nu_l$, $i \le l)$ for given $y = y^l_{j_l} \in E_l$ follows from (2).

For simplicity let $\operatorname{dom}\nu_l = [0, m_l]$ and $m_l := |\mathcal{P}_l| - 1$ $(l \in \mathbb{N})$, with $m = m_l$ determined by l. To complete the construction, there is some freedom in choice of ν_l. We will ensure each $f \in \mathcal{F}$ has a computable ρ-name; the $(l+1)$-block $j^{(q)}_0, \ldots, j^{(q)}_l$ relevant to f can be picked out of ν-name a based on this. More rigourously, fix choice function $c'' :\subseteq \mathcal{F} \times \mathbb{N}^3 \to \mathbb{N}$ s.t.

$$d(fy^l_j, y^k_{c''(f,l,k,j)}) < \theta_k \quad \text{for all } (f,l,k,j) \in \operatorname{dom} c'' := \{(f,l,k,j) \mid f \in \mathcal{F} \wedge l < k \wedge 1 \le j \le r_l\}.$$

For any fixed k and $f \in \mathcal{F}$, from $p \in \rho^{-1}\{f\}$ and $a \in \operatorname{dom}\nu$ we will be able to compute

$$u := \begin{cases} j^{(p_l)}_k, & \text{if } k \le l, \\ c''(f,l,k,j_l), & \text{if } l < k, \end{cases}$$

and observe $d(fy^l_{j_l}, y^k_u) < \theta_k$; this computation is possible since $(\{f\} \times \mathbb{N} \times \{k\} \times \mathbb{N}) \cap \operatorname{dom} c''$ is finite.

To define ν_l $(l \in \mathbb{N})$, proceed in stages; at the end of stage i we will have each ν_l defined on an interval $[0, m_{l,i}]$ with

$$\nu_l[0, m_{l,i}] = \{[f_j]_{\sim_l} \mid j \leq i\} \qquad \text{for all } l \in \mathbb{N}. \quad (3)$$

In stage 0 we define $m_{l,0} := 0$ and declare $0 \in \nu_l^{-1}\{[f_0]_{\sim_l}\}$ for all l; also let $l_0 := 0$ and $p^{(0)} := 0^\omega$ $(\in \rho^{-1}\{f_0\})$.

At stage $i+1$ let $l_{i+1} := \inf\{l \in \mathbb{N} \mid (\forall j \leq i)(f_{i+1} \not\sim_l f_j)\}$ $(i \in \mathbb{N})$. For each $l < l_{i+1}$ we have $f_{i+1} \in \cup_{j \leq i}[f_j]_{\sim_l}$, so $(\exists p_l^{(i+1)} \leq m_{l,i})(\nu_l(p_l^{(i+1)}) = [f_{i+1}]_{\sim_l})$, and we leave ν_l unmodified, $m_{l,i+1} := m_{l,i}$. For $l \geq l_{i+1}$ we have $[f_{i+1}]_{\sim_l} \notin \{[f_j]_{\sim_l} \mid j \leq i\} = \nu_l[0, m_{l,i}]$, so let $p_l^{(i+1)} := m_{l,i+1} := m_{l,i} + 1$ and $\nu_l(m_{l,i+1}) := [f_{i+1}]_{\sim_l}$.

In either case $(3)|_{i+1}$ holds (by inspection), computability of $(m_{l,i+1})_l \in \mathbb{N}^{\mathbb{N}}$ holds by induction, and the computability of $p^{(i+1)} \in \rho^{-1}\{f_{i+1}\}$ will follow once we show $l_{i+1} < \infty$. Suppose that $l_{i+1} = \infty$; then $(\forall l)(\exists j \leq i)(f_{i+1} \sim_l f_j)$. By the pigeonhole principle and the fact $(\mathcal{P}_l)_l$ are increasingly fine, we get $(\exists j \leq i)(\forall l)(f_{i+1} \sim_l f_j)$, so $f_{i+1} = f_j$ since $(\mathcal{P}_l)_l$ is generating. This contradicts injectivity of $e \mapsto f_e$.

Finally, since $\{f_e \,\big|\, e \in \mathbb{N}\} = \mathcal{F}$, $(\forall i)(3)|_i$ implies $\nu_l(\cup_i[0, m_{l,i}]) = \mathcal{P}_l$ for each l. It is clear by construction ν_l injective, so $\sup_i m_{l,i}$ finite, and $\sup_i m_{l,i} = m_l = |\mathcal{P}_l| - 1$ as previously assumed. $\square$

Clear how to check compactness of ν w.r.t. $(B(y_j^k; \theta_k))_{j=1}^{r_k}$: use choice function $c^+ :\subseteq \mathbb{N}^3 \to \mathbb{N}$ with $d(y_j^l, y_{c^+(l,k,j)}^k) < \theta_k$ for all $(l,k,j) \in \mathrm{dom}\, c^+ := \{(l,k,j) \,\big|\, l < k, 1 \leq j \leq r_l\}$, and from $a \in \mathrm{dom}\,\nu$ compute

$$u := \big(j_k,\ \text{if } l \geq k;\ c^+(l,k,j_l),\ \text{if } l < k\big)\,;$$

this uses finiteness of $\mathbb{N} \times \{k\} \times \mathbb{N}) \cap \mathrm{dom}\, c^+$.

If $\mathrm{id}_X \in \mathcal{F}$, this argument (and data $j_0, \ldots, j_{l-1}$ in names $a \in \mathrm{dom}\,\nu$) can be omitted. Despite these good properties, several reasons to simplify or generalise above proof.

Examples for $\sim_l$

Consider equivalence relations $\sim_l$ ($l \in \mathbb{N}$) defined from choice function $\tilde{c}$ as follows: fix $\tilde{c} :\subseteq \mathcal{F} \times \mathbb{N}^3 \to \mathbb{N}$ such that

$$d(fy^l_j, y^i_{\tilde{c}(f,l,i,j)}) < \theta_i \quad \text{for all } (f,l,i,j) \in \operatorname{dom} \tilde{c} := \{(f,l,i,j) \,\big|\, i \le l, 1 \le j \le r_l\},$$

define $f \sim_l f'$ if $\tilde{c}(f,\cdot), \tilde{c}(f',\cdot)$ agree on $([0,l] \times \mathbb{N}^2) \cap \operatorname{dom} \tilde{c}(f,\cdot)$. Such $\sim_l$ are increasingly fine, have $\mathcal{F}/{\sim_l}$ finite and (2) holds (definition of ν can further be 'simplified' by requiring strictly $j^{(q)}_i = \tilde{c}(f,l,i,j_l)$ for all $i \le l$ whenever $f \in \nu_l(q)$, $q \le m_l$).

If $f, f' \in \mathcal{F}$ distinct over $\cup_k E_k$, pick k, $x \in E_k$ s.t. $f(x) \neq f'(x)$, and w.l.o.g. assume $(\forall l)(E_l \subseteq E_{l+1})$. For $l \ge k$ s.t. $2\theta_l < d(fx, f'x)$ and $j \in [r_l]$ s.t. $y^l_j = x$ we must have $\tilde{c}(f,l,l,j) \neq \tilde{c}(f',l,l,j)$, hence $f \not\sim_l f'$. So the separating condition may be replaced by

$$(\forall f, f' \in \mathcal{F})(f \neq f' \implies f|_{\cup_k E_k} \neq f'|_{\cup_k E_k}),$$

and $\cup_k E_k$ chosen as any dense countable subset of $g^{-1} \operatorname{im} \lambda$ (while assuming $(\forall l)(E_l \subseteq E_{l+1})$).

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